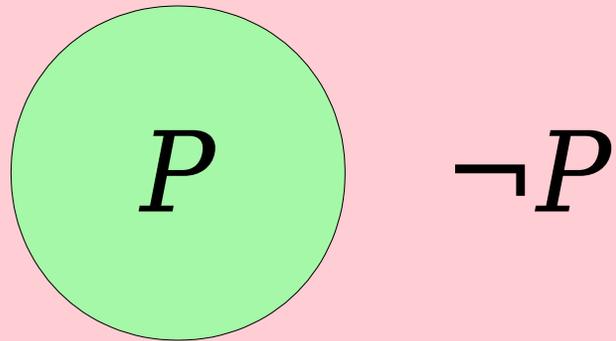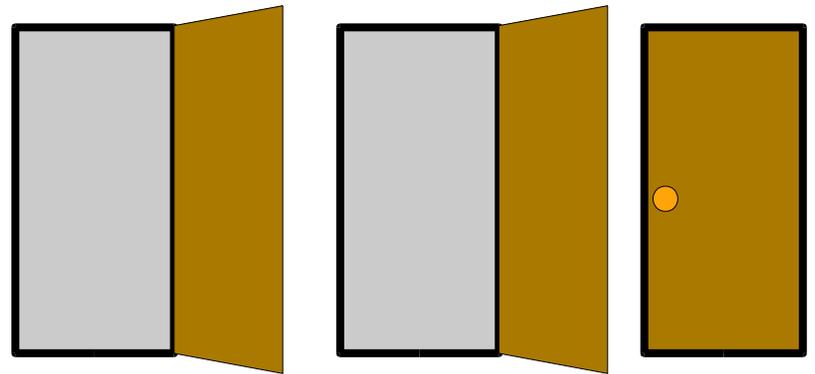# Indirect Proofs

Indirect Proofs
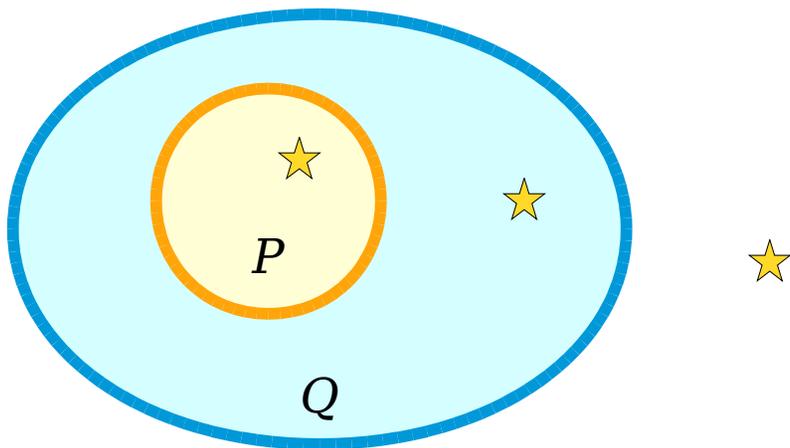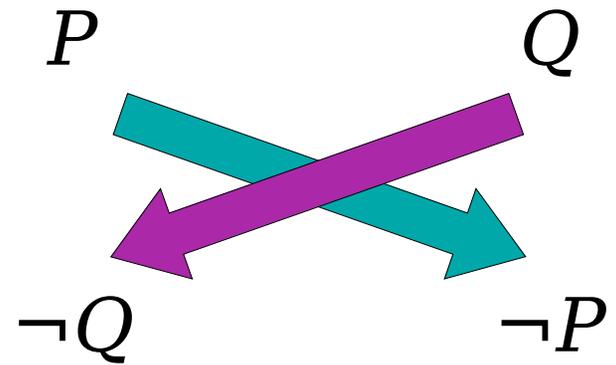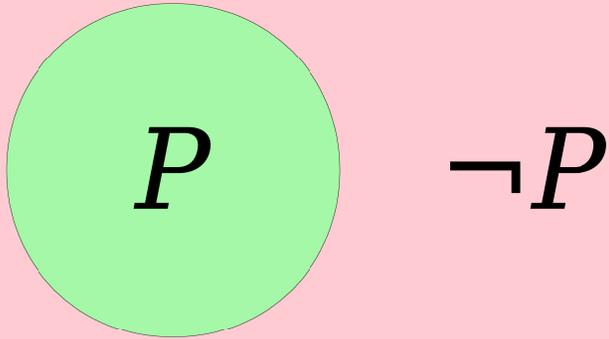
A Story in Four Acts

Logical Negation

Proof by Contradiction

Logical Implication

Proof by Contrapositive

Logical Negation

P

$\neg P$

Proof by Contradiction

Logical Implication

P

Q

Proof by Contrapositive

P

Q

$\neg Q$

$\neg P$
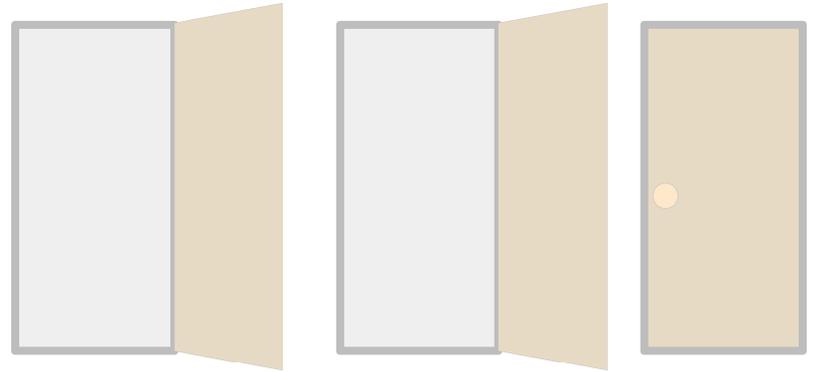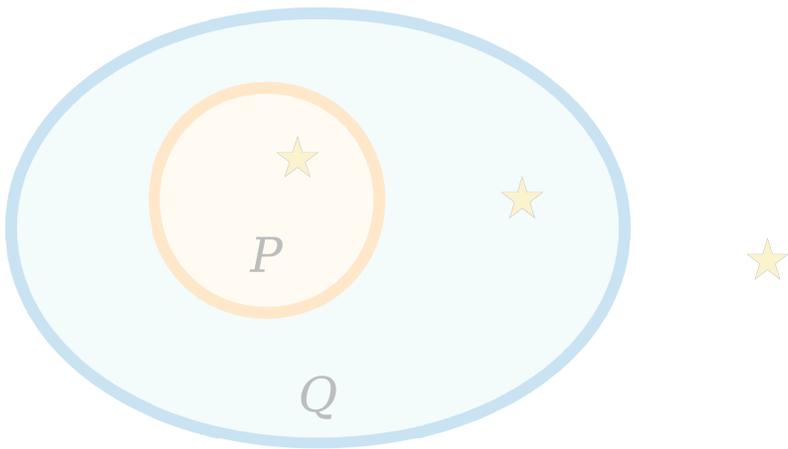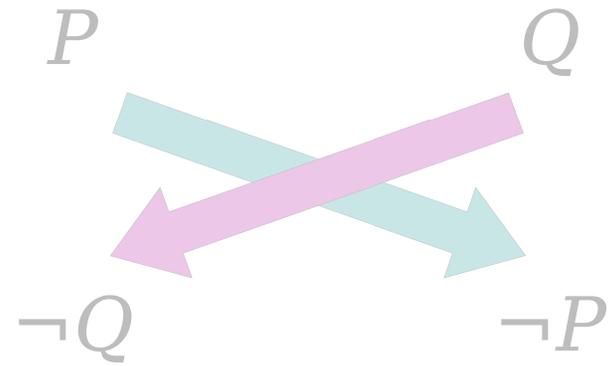
Logical Negation

Negations

- A ***proposition*** is a statement that is either true or false.

- Some examples:

  - If n is an even integer, then n^2 is an even integer.

  - $\varnothing = \mathbb{R}$.

- The ***negation*** of a proposition X is a proposition that is true when X is false and is false when X is true.

- For example, consider the proposition "it is snowing outside."

  - Its negation is "it is not snowing outside."

  - Its negation is *not* "it is sunny outside." ⚠

  - Its negation is *not* "we're in the Bay Area." ⚠

How do you find the negation of a statement?

"All My Friends Are Taller Than Me"

Me
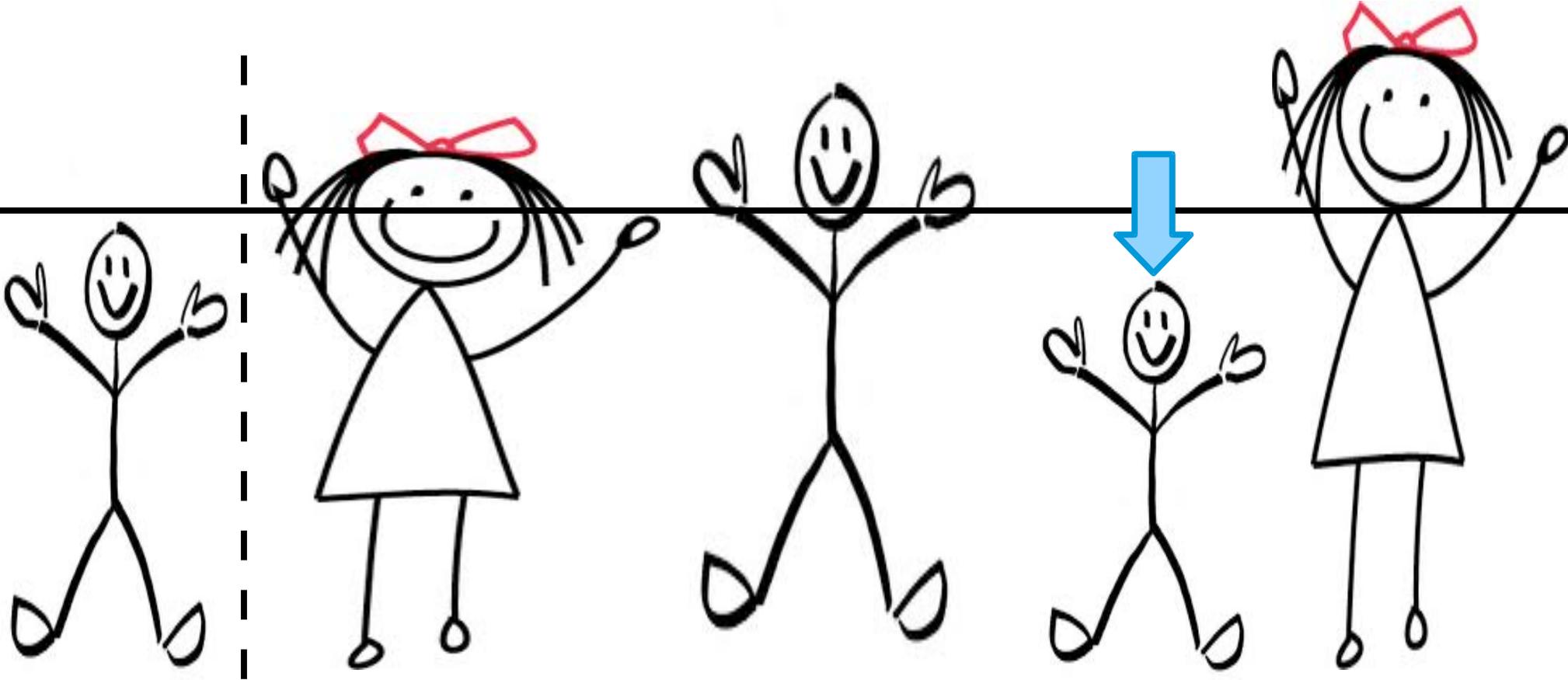
My Friends

The negation of the **universal** statement

Every *P* is a *Q*

is the **existential** statement

There is a *P* that is not a *Q*.

The negation of the **_universal_** statement

For all _x_, _P_(_x_) is true.

is the **_existential_** statement

There exists an _x_ where _P_(_x_) is false.

"Some Friend Is Shorter Than Me"

Me
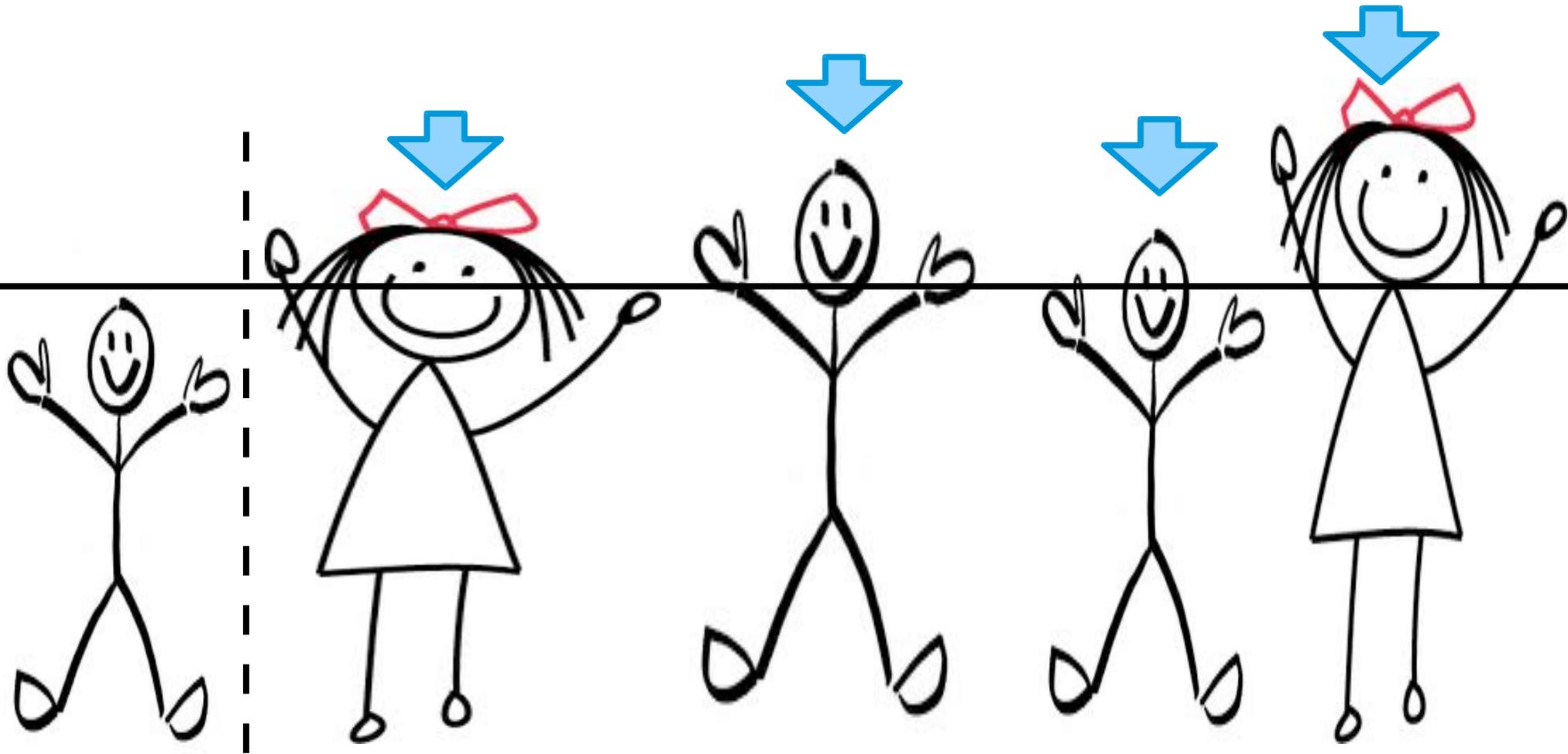
My Friends

The negation of the **existential** statement

There exists a P that is a Q

is the **universal** statement

Every P is not a Q.

The negation of the **existential** statement

There exists an x where $P(x)$ is true

is the **universal** statement

For all x, $P(x)$ is false.

Your Turn!

- What's the negation of the following statement?

**"Every brown dog
loves every orange cat."**

Your Turn!

- What's the negation of the following statement?

"Every brown dog loves every orange cat."

- Answer:

"There is a brown dog that doesn't love some orange cat"
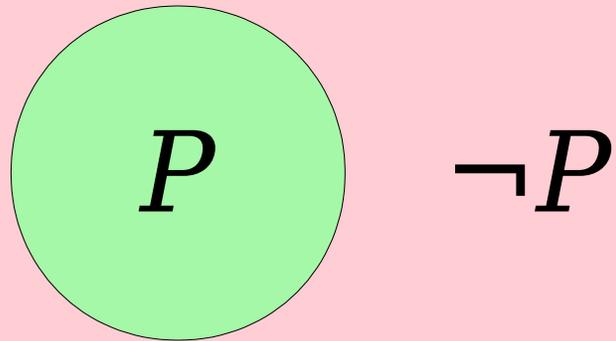
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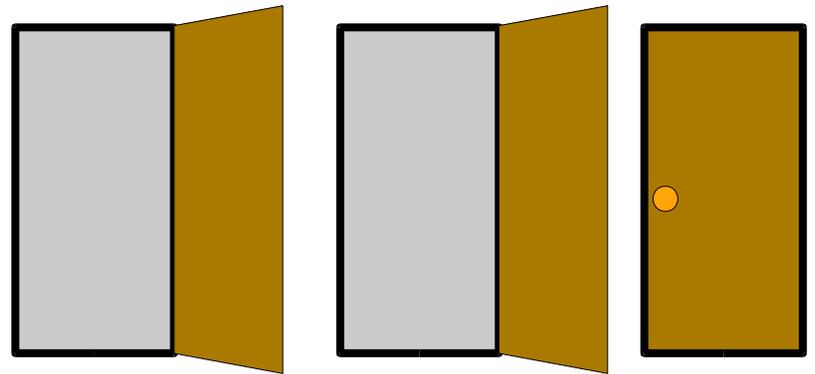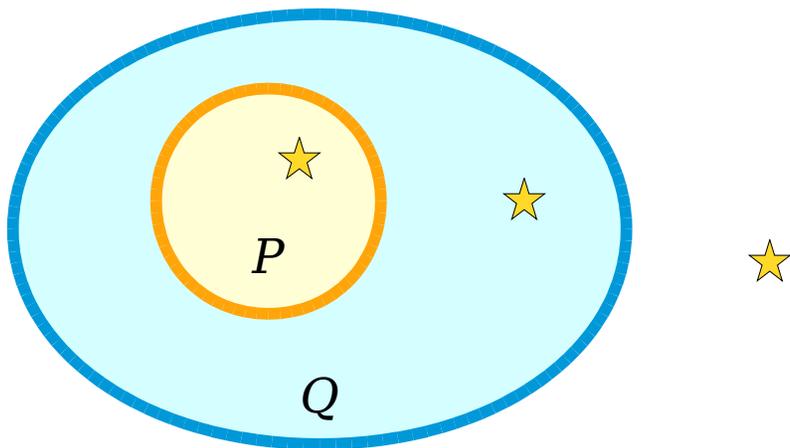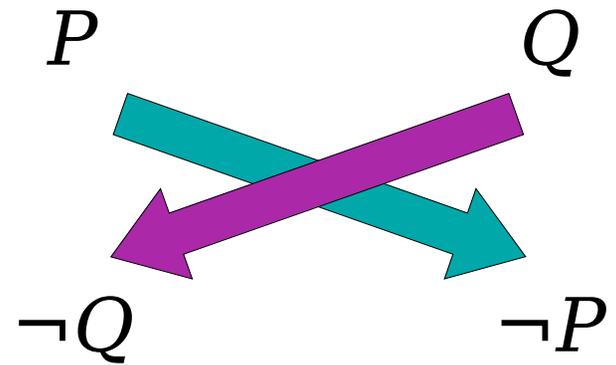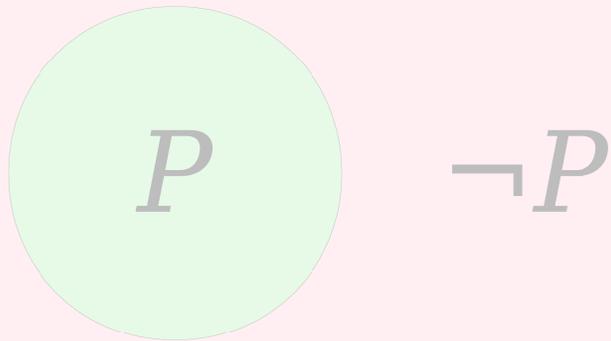
Proof by Contradiction

Logical Implication

Proof by Contrapositive

Logical Negation

P $\neg P$

Proof by Contradiction

Logical Implication

P

Q

Proof by Contrapositive

P Q

$\neg Q$ $\neg P$
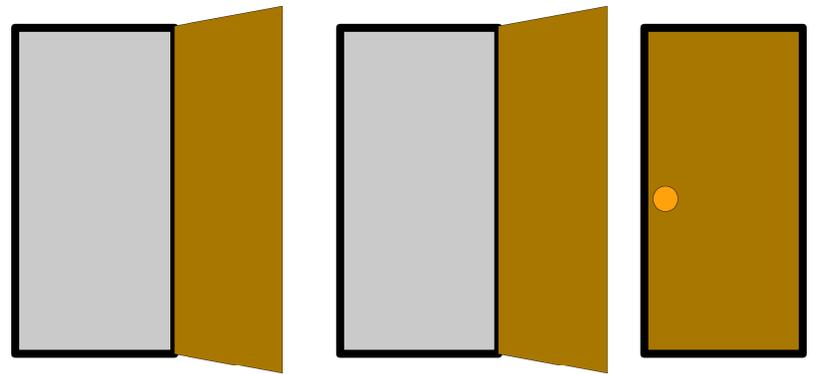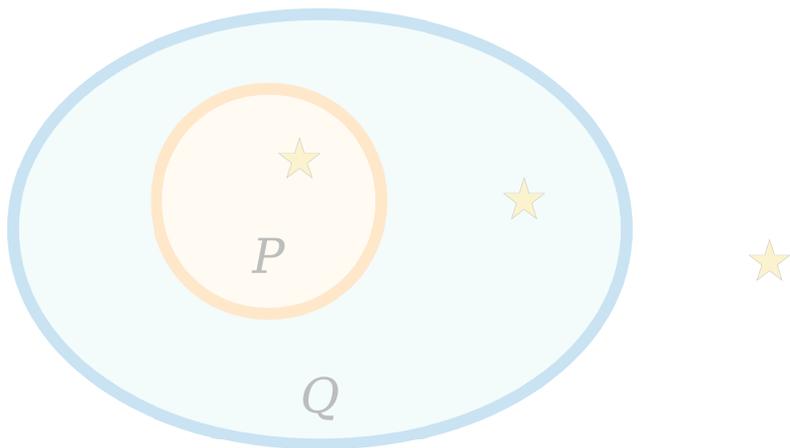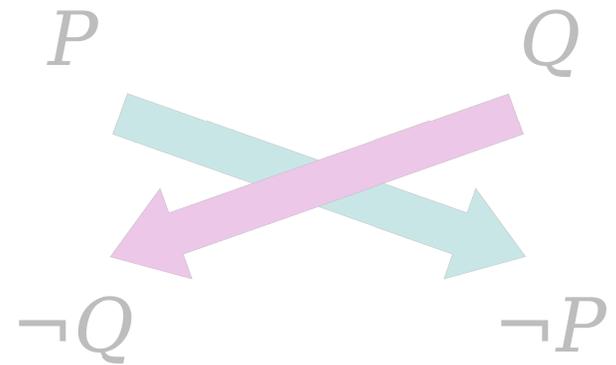
Proof by Contradiction

There's something hidden behind one of these doors. Which door is it hidden behind?

Door 1

Door 2

Door 3
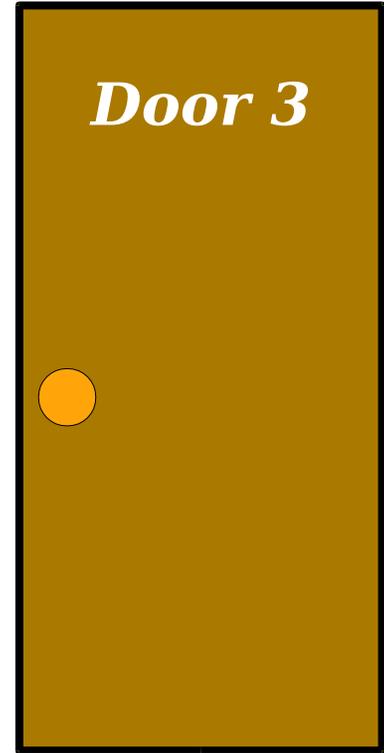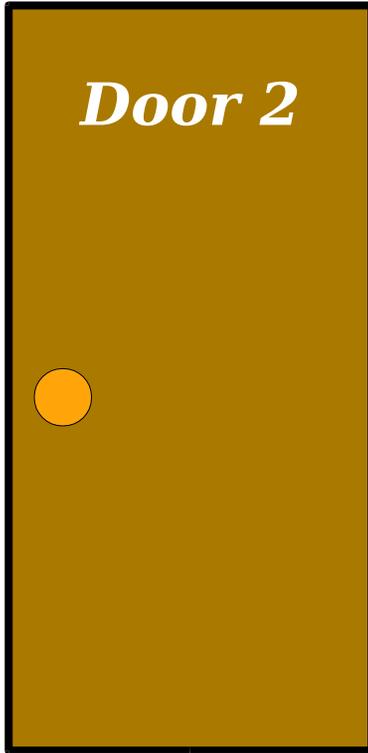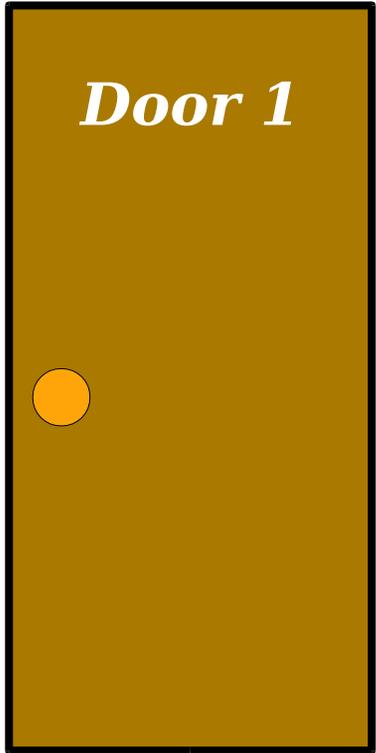
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Door 3

Even without opening this door, we know whatever is hidden has to be here.
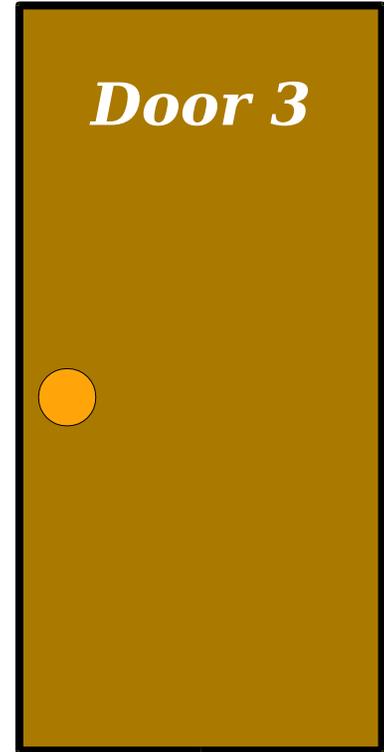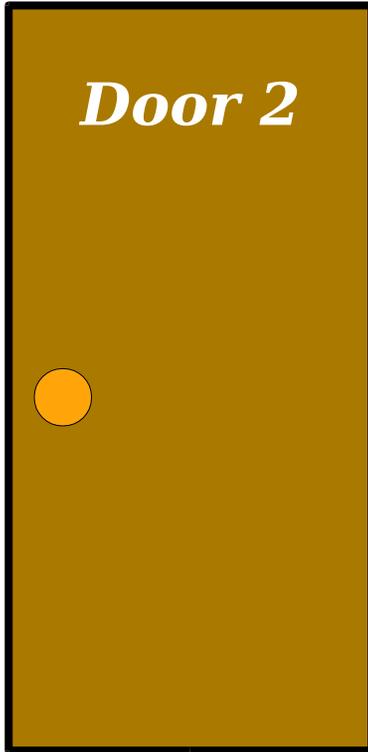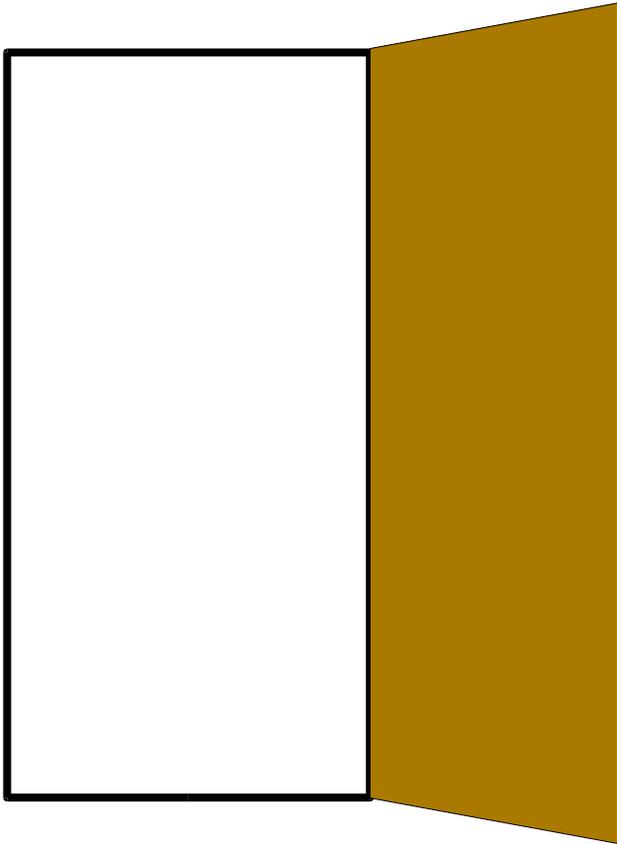
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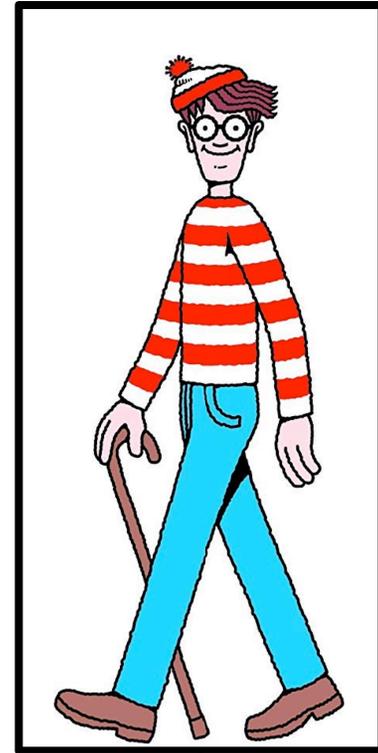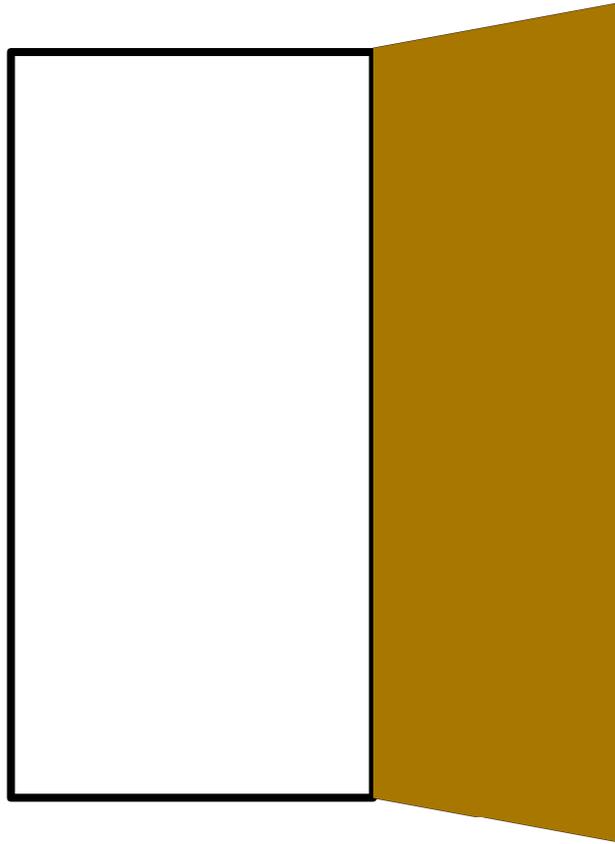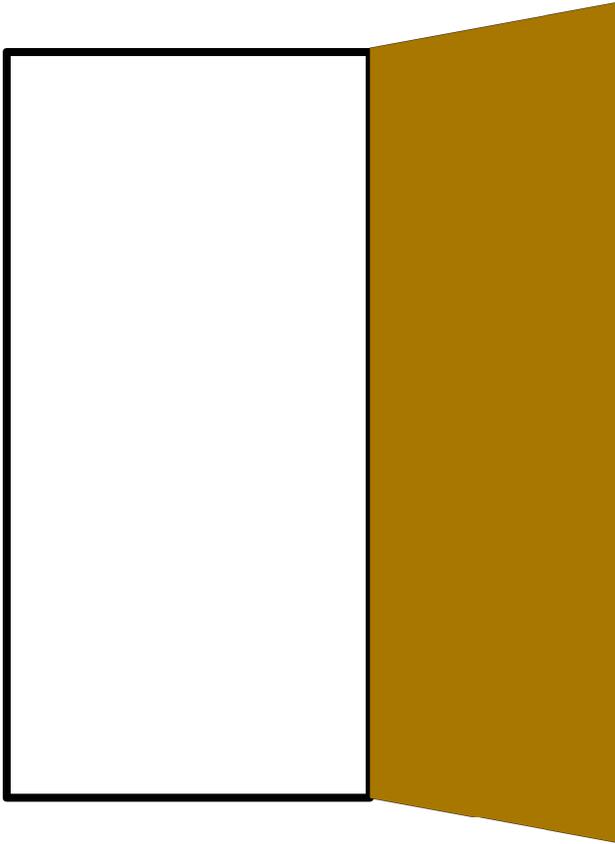
Even without opening this door, we know whatever is hidden has to be here.

Every statement in mathematics is either true or false.
*If statement P is **not false**, what does that tell you?*

The Door
of
Truth

The Door
of
Falsity

Every statement in mathematics is either true or false. If statement P is not false, what does that tell you?

The Door
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Even without opening this door,
we know **P** has to be here.

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P
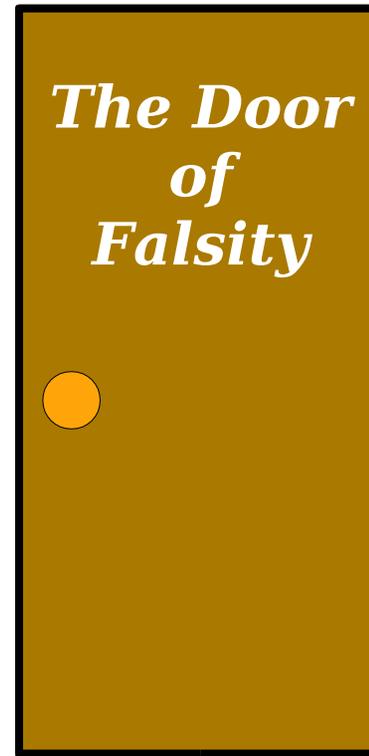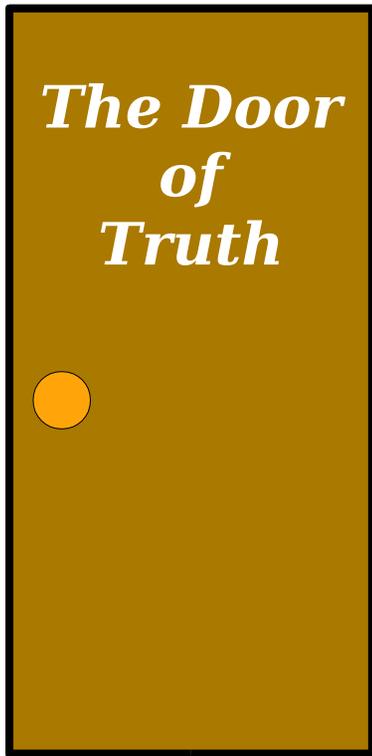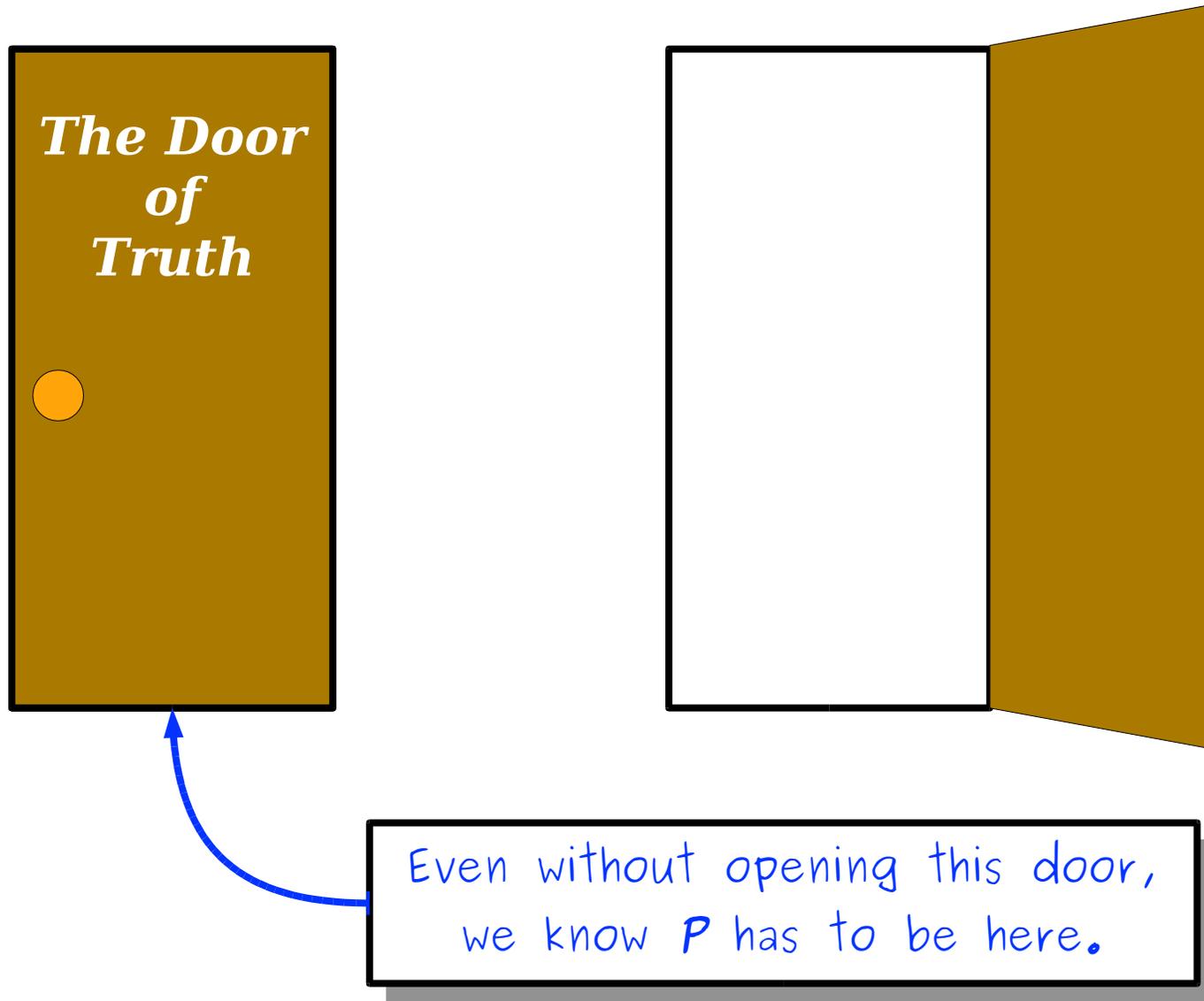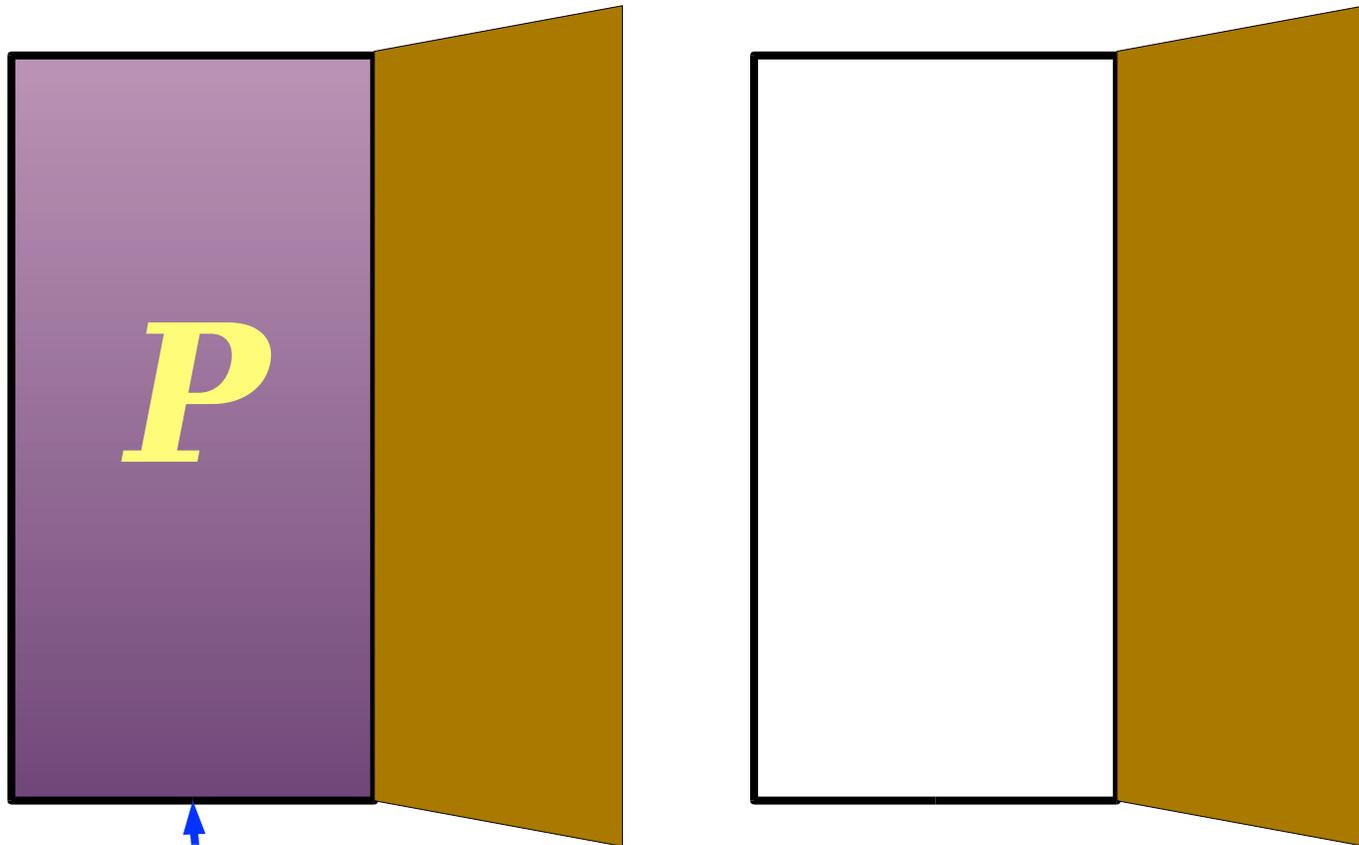
Even without opening this door, we know P has to be here.

A *proof by contradiction* shows that some statement P is true by showing that P isn't false.

Proof by Contradiction

- ***Key Idea:*** Prove a statement P is true by showing that it isn't false.

- First, assume that P is false. The goal is to show that this assumption is silly.

- Next, show this leads to an impossible result.

  - For example, we might have that $1 = 0$, that $x \in S$ and $x \notin S$, that a number is both even and odd, etc.

- Finally, conclude that since P can't be false, we know that P must be true.

An Example: *Set Cardinalities*

Set Cardinalities

- We've seen sets of many different cardinalities:

  - $|\varnothing| = 0$

  - $|\{1, 2, 3\}| = 3$

  - $|\{\, n \in \mathbb{N} \mid n < 137 \,\}| = 137$

  - $|\mathbb{N}| = \aleph_0.$

  - $|\wp(\mathbb{N})| > |\mathbb{N}|$

- These span from the finite up through the infinite.

- *Question:* Is there a "largest" set? That is, is there a set that's bigger than every other set?

Theorem: There is no largest set.

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Proof:

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Proof:

To prove this statement by contradiction, we're going to assume its negation.

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One option: "there *is* a largest set."

Theorem: There is no largest set.

Proof: Assume for the sake of contradiction that there is a largest set; call it S.

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1. that this is a proof by contradiction, and
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This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

We've reached a contradiction, so our assumption must have been wrong. Therefore, there is no largest set. ▪

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Another Example

Latin Squares

- A ***Latin square*** is an $n \times n$ grid filled with the numbers $1, 2, \ldots, n$ such that every number appears in each row and each column exactly once.

1	2	3
2	3	1
3	1	2

1	3	4	2
4	2	1	3
2	1	3	4
3	4	2	1

1	3	5	2	4
3	1	4	5	2
4	5	2	3	1
5	2	1	4	3
2	4	3	1	5

3	2	1	4	5	6
2	4	6	1	3	5
5	6	4	3	2	1
4	1	5	2	6	3
6	3	2	5	1	4
1	5	3	6	4	2

Latin Squares

- A **Latin square** is an $n \times n$ grid filled with the numbers $1, 2, \ldots, n$ such that every number appears in each row and each column exactly once.

- The **main diagonal** of a Latin square runs from the top-left corner to the bottom-right corner.

- A Latin square is **symmetric** if the numbers are symmetric across the main diagonal.

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1	5	3	6	4	2

Latin Squares

- Notice anything about what's on the main diagonals of these symmetric Latin squares?

- ***Theorem:*** Every odd-sized symmetric Latin square has every number 1, 2, ..., *n* on its main diagonal.

1	2	3
2	3	1
3	1	2

1	2	3	4	5
2	5	4	1	3
3	4	2	5	1
4	1	5	3	2
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Theorem: Every symmetric Latin square of odd size $n \times n$ has each
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Theorem: Every symmetric Latin square of odd size $n \times n$ has each
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Proof:

What is the negation of the theorem?

***Every symmetric Latin square of odd size n × n has
each of the numbers 1, 2, ..., n on its main diagonal.***

Theorem: Every symmetric Latin square of odd size $n \times n$ has each
 of the numbers $1, 2, ..., n$ on its main diagonal.

Proof:

What is the negation of the theorem?

Every symmetric Latin square of odd size n × n has each of the numbers 1, 2, ..., n on its main diagonal.

One option:

There is a symmetric Latin square of odd size n × n that does not have one of the numbers 1, 2, ..., n on its main diagonal.

Theorem: Every symmetric Latin square of odd size $n \times n$ has each
of the numbers 1, 2, ..., n on its main diagonal.

Proof: Assume for the sake of contradiction that there is a
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Combining these results, we see that $n = 2k$.

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Combining these results, we see that $n = 2k$. This means that n is even, contradicting the fact that n is odd.

Theorem: Every symmetric Latin square of odd size $n \times n$ has each
    of the numbers 1, 2, …, n on its main diagonal.

Proof: Assume for the sake of contradiction that there is a
    symmetric Latin square of odd size $n \times n$ that does not have one
    of the numbers 1, 2, 3, …, n on its main diagonal. Call the
    missing number r.

Let k be the number of times r appears above the main diagonal.
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Combining these results, we see that $n = 2k$. This means that n is even, contradicting the fact that n is odd. We've reached a contradiction, so our assumption was wrong. Therefore, all symmetric Latin squares of odd size $n \times n$ have each of the numbers 1, 2, ..., and n on their main diagonals.

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missing number r.

The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

Independently, we know that r appears n times in the Latin
square, once for each of its n rows.

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Time-Out for Announcements!

Problem Set One

- Problem Set One goes out today. It's due next Friday at 3:00PM.

  - Explore the language of set theory and better intuit how it works.

  - Learn more about the structure of mathematical proofs.

  - Write your first "freehand" proofs based on your experiences.

- As always, reach out if you have any questions!

Office Hours

- It is **_completely normal_** in this class to need to get help from time to time.

- Feel free to ask clarifying and conceptual questions on EdStem.

- Need more structured help? We have office hours! Feel free to stop on by.

  - Check out the online "Guide to Office Hours" for more information about how our office hours system works.

  - The OH calendar will soon be available on the course website.

- Office hours start this Monday.

Readings for Today

- On the course website we have some information you should look over.

- First is the *Proofwriting Checklist*. It contains information about style expectations for proofs. We'll be using this when grading, so be sure to read it over.

- Next is the *Guide to Office Hours*, which talks about how our office hours work and how to make the most effective use of them.

- Finally is the *Guide to LaTeX*, which explains how to use LaTeX to typeset your problem sets in a way that's so beautiful it will bring tears to your eyes.
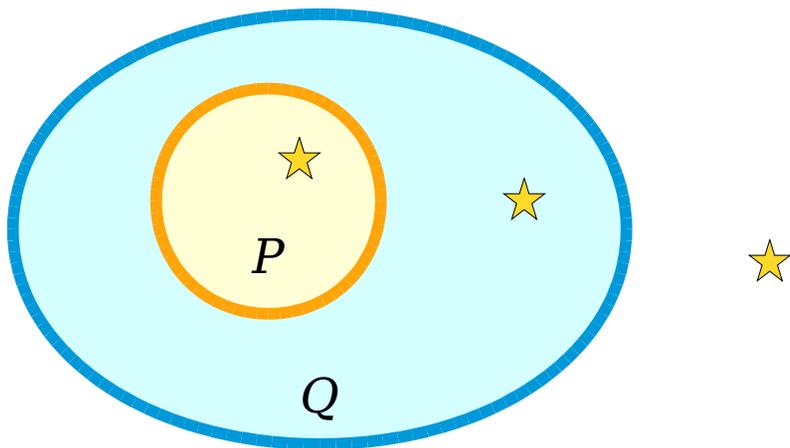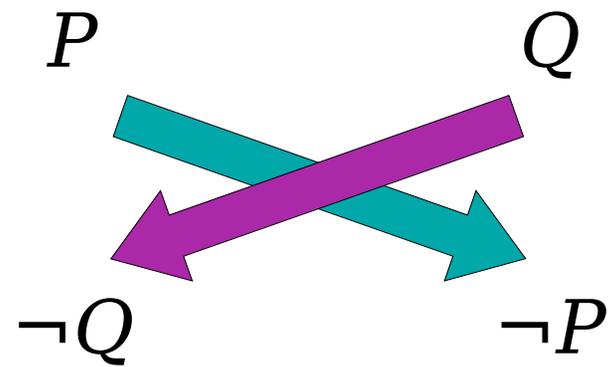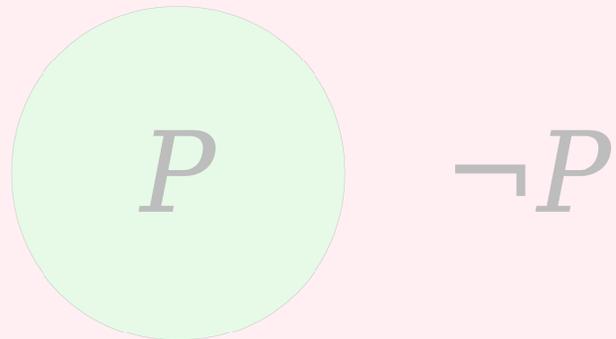
Back to CS103!
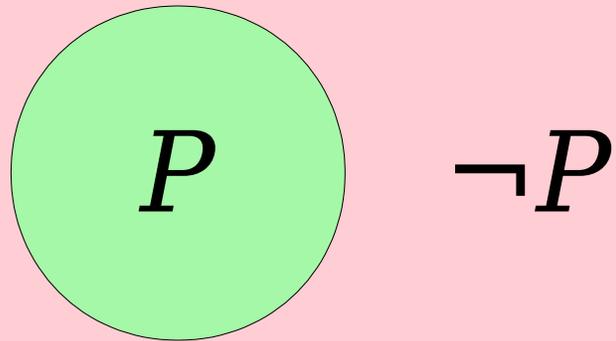
Logical Negation
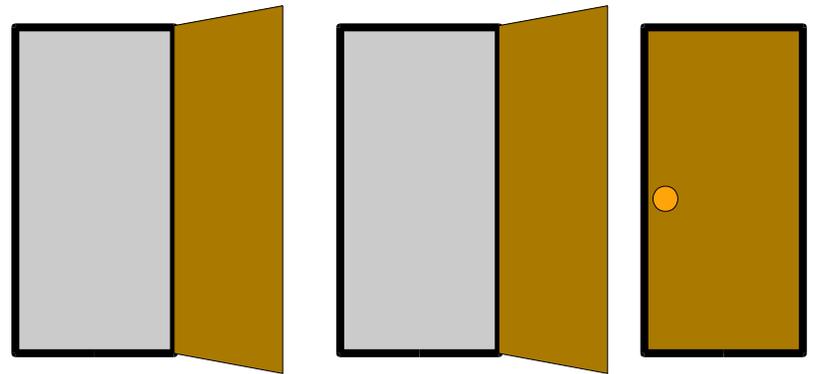
Proof by Contradiction
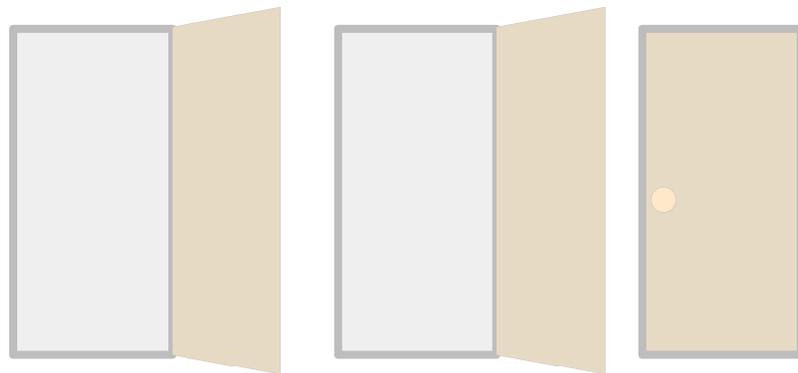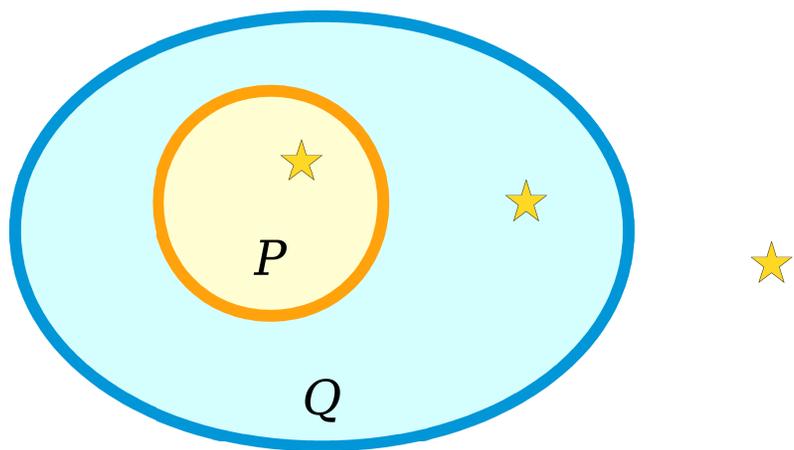
Logical Implication

Proof by Contrapositive

Logical Negation

Proof by Contradiction

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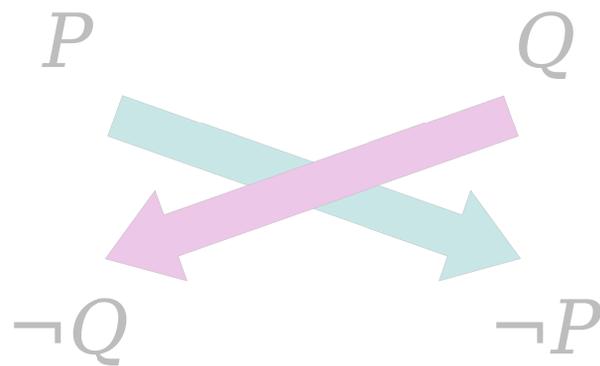
Proof by Contrapositive

If n is an even integer, then n^2 is an even integer.

An ***implication*** is a statement of the form
"If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

This part of the implication is called the *antecedent*.

This part of the implication is called the *consequent*.

An ***implication*** is a statement of the form "If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

If m and n are odd integers, then $m+n$ is even.

If you like the way you look that much,
then you should go and love yourself.

An ***implication*** is a statement of the form
"If P is true, then Q is true."

What Implications Mean

"If there's a rainbow in the sky, then it's raining somewhere."

- In mathematics, implication is directional.

  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.

- In mathematics, implications only say something about the consequent when the antecedent is true.

  - If there's no rainbow, it doesn't mean there's no rain.

- In mathematics, implication says nothing about causality.

  - Rainbows do not cause rain. 😃
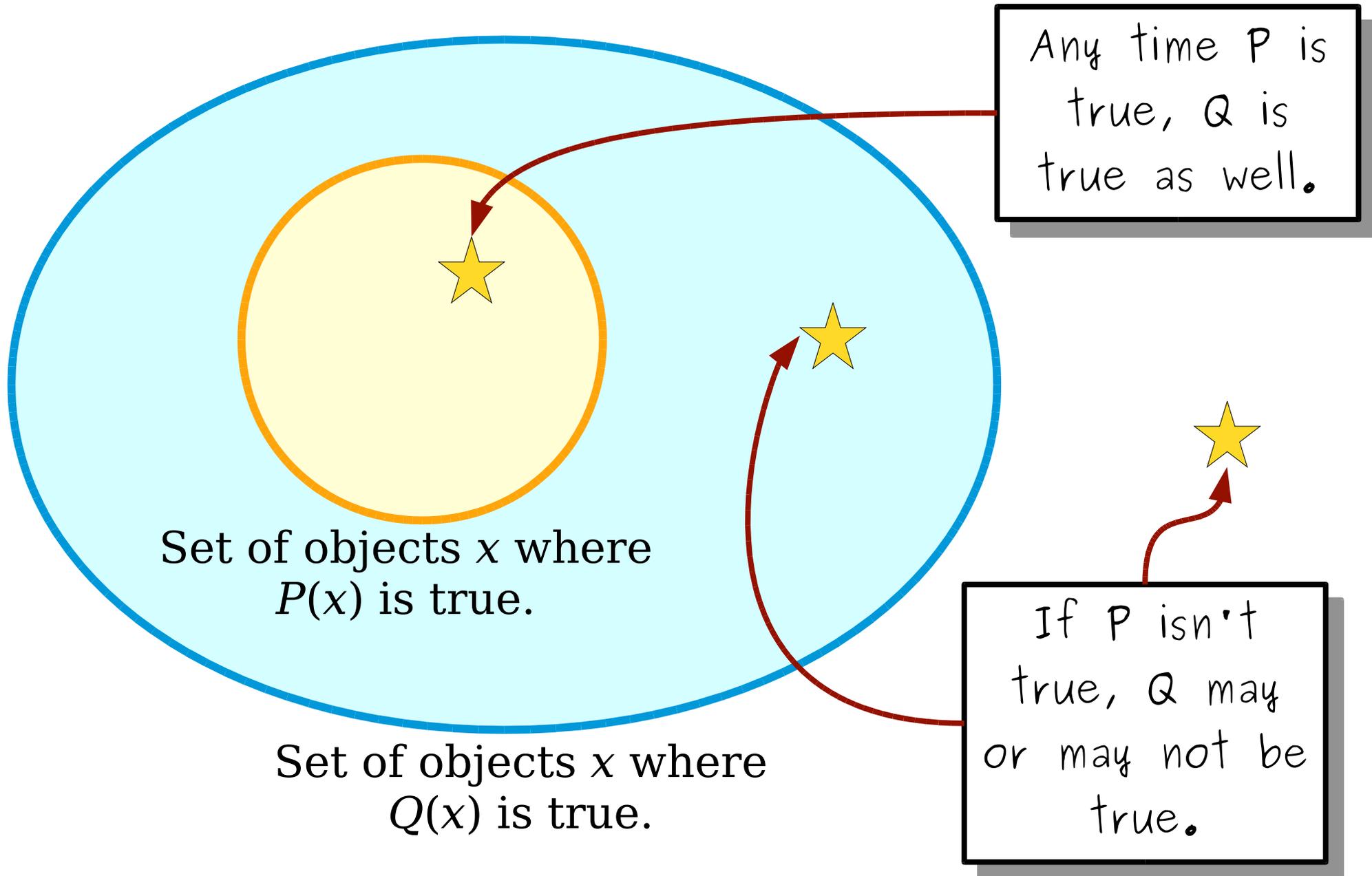
What Implications Mean

- In mathematics, a statement of the form

  For any *x*, if *P(x)* is true, then *Q(x)* is true

  means that any time you find an object *x* where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same *x*).

- There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.

# Implication, Diagrammatically

Set of objects *x* where
P(x) is true.

Set of objects *x* where
Q(x) is true.

Any time P is true, Q is true as well.

If P isn't true, Q may or may not be true.
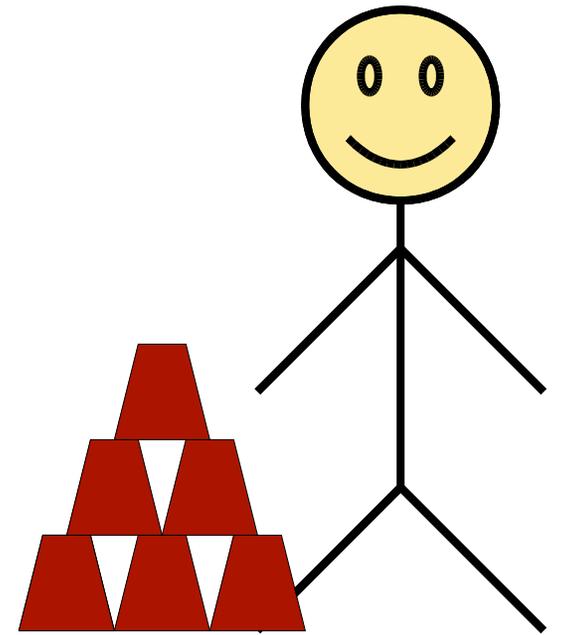
How do you negate an implication?

Story Time!

Ancient Contract:

If Nanni pays money to Ea-Nasir, then Ea-Nasir will give Nanni quality copper ingots.

I'm going to complain about this! (That's a hyperlink. Click it.)
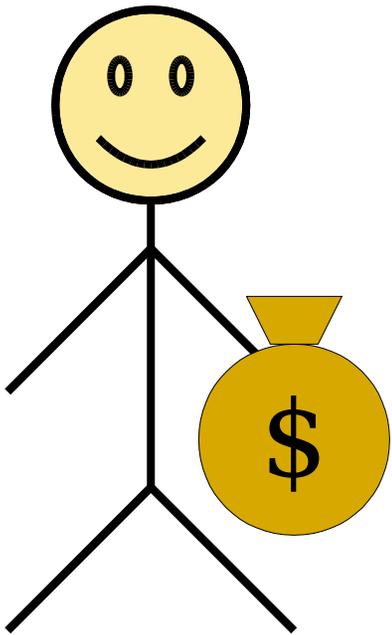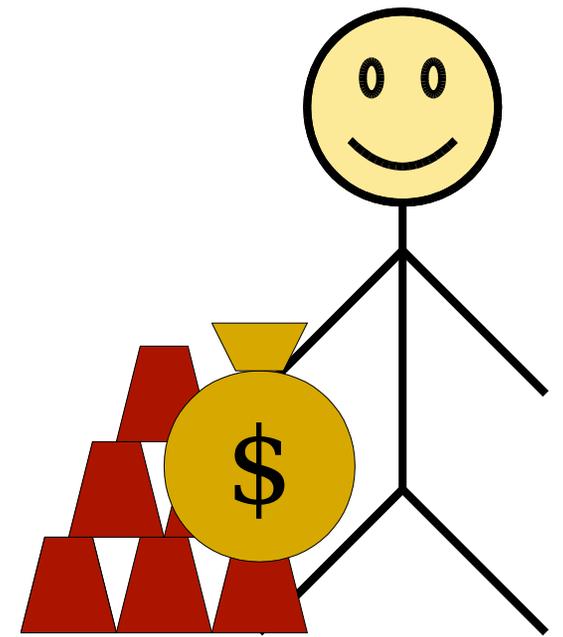
Nanni

Ea-Nasir

Question: What has to happen for this contract to be broken?
Answer: Nanni pays Ea-Nasir and doesn't get quality copper ingots.

The negation of the statement

**"For any *x*, if *P*(*x*) is true,
then *Q*(*x*) is true"**

is the statement

**"There is at least one *x* where
P(*x*) is true and *Q*(*x*) is false."**

*The negation of an implication
is not an implication!*

The negation of the statement

**"For any *x*, if *P(x)* is true,
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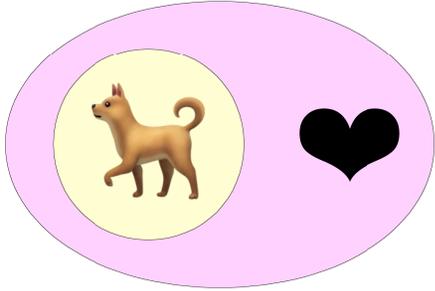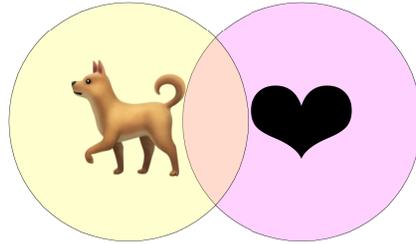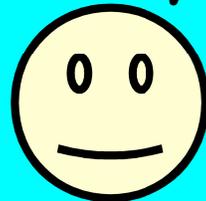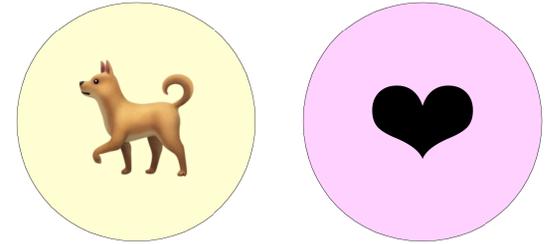
How to Negate Universal Statements:

"For all x, $P(x)$ is true"

becomes

"There is an x where $P(x)$ is false."

How to Negate Existential Statements:

"There exists an x where $P(x)$ is true"

becomes

"For all x, $P(x)$ is false."

How to Negate Implications:

"For every x, if $P(x)$ is true, then $Q(x)$ is true"

becomes

"There is an x where $P(x)$ is true and $Q(x)$ is false."
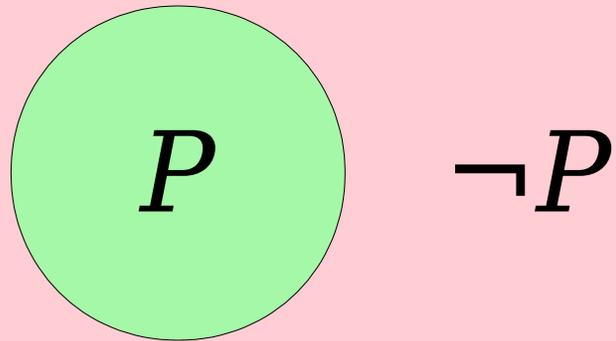
Logical Negation
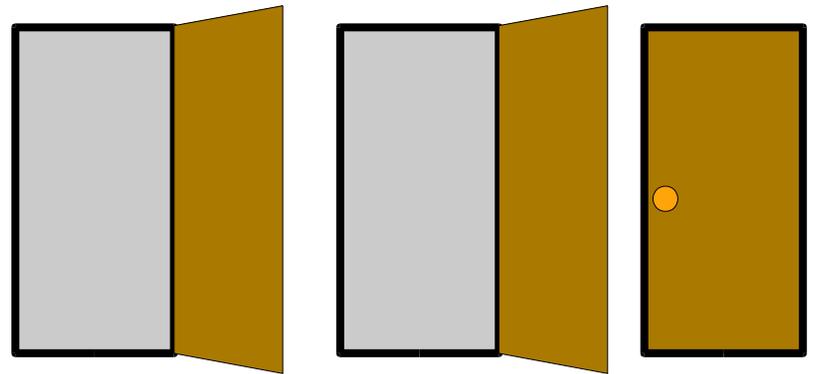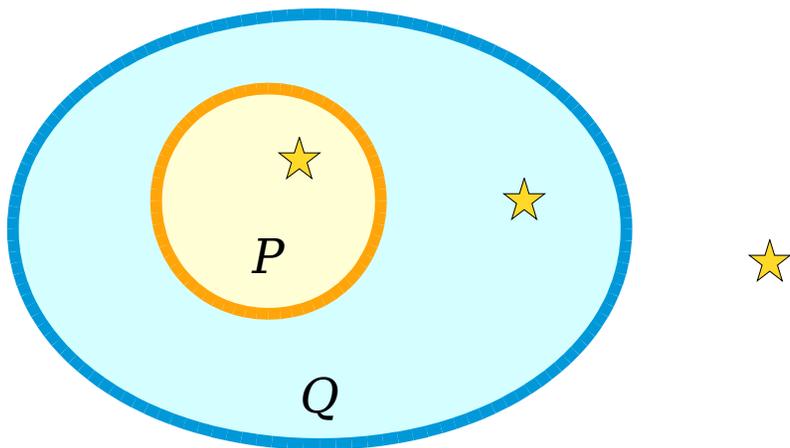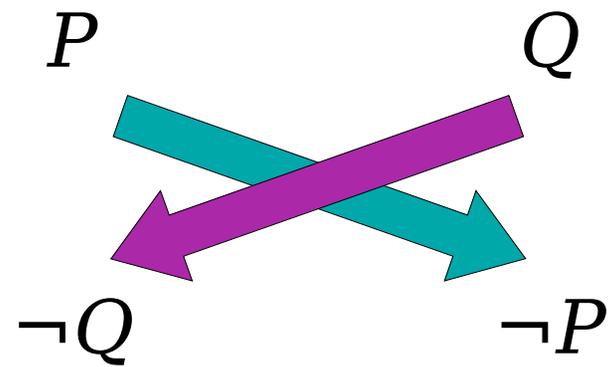
Proof by Contradiction

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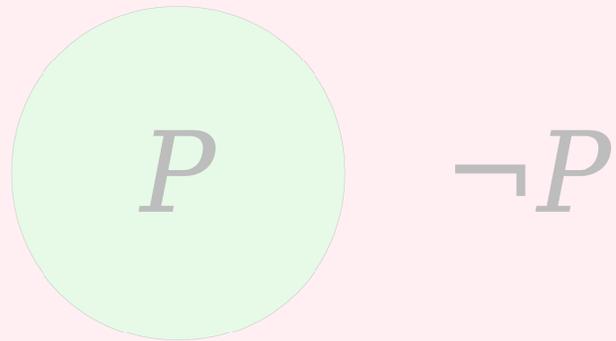
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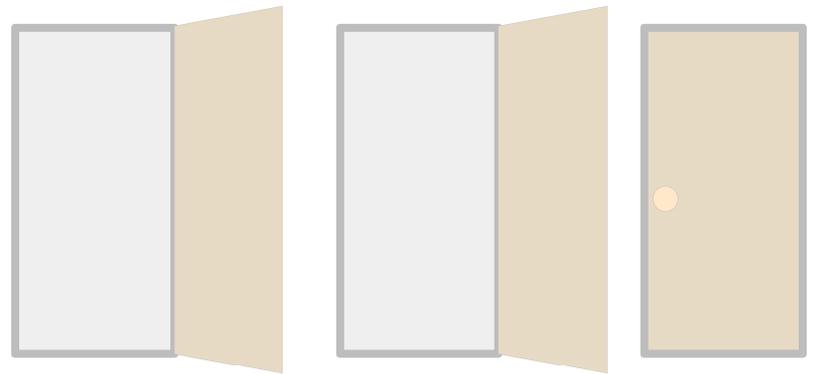
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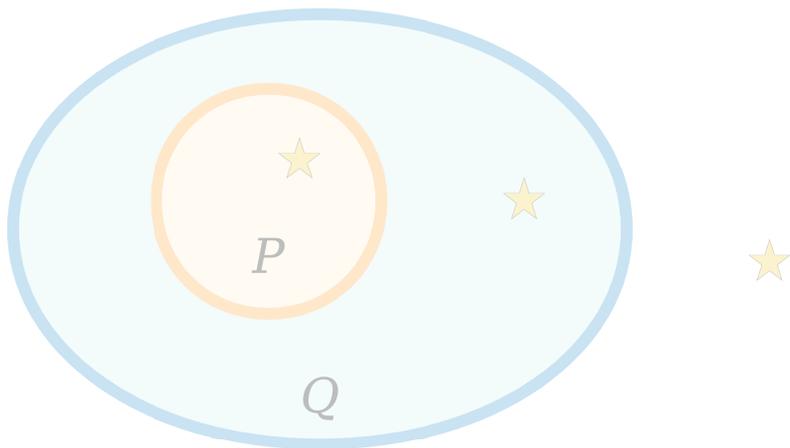
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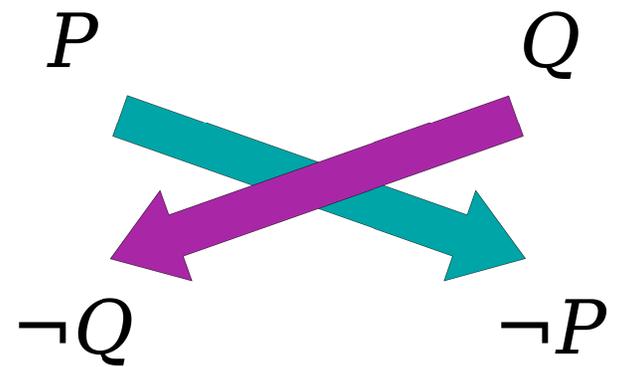
Proof by Contrapositive

Anything inside this inner bubble is also inside the outer bubble.

Anything outside this outer bubble is outside the inner bubble.

Times where P is true

Times where Q is true
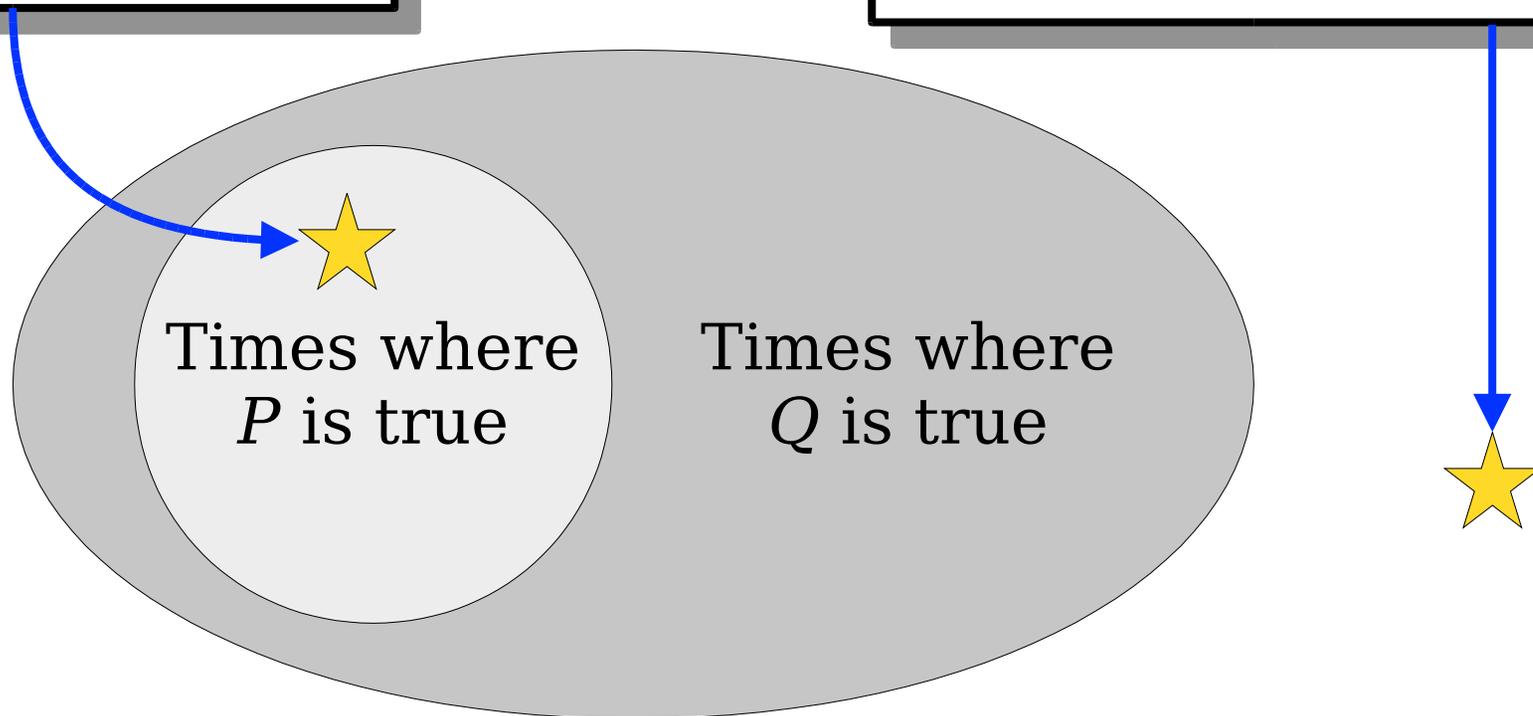
If P is true, then Q is true.

If Q is false, then P is false.

The Contrapositive

- The ***contrapositive*** of the implication

  If *P* is true, then *Q* is true

  is the implication

  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

  If it's a puppy, then I love it.
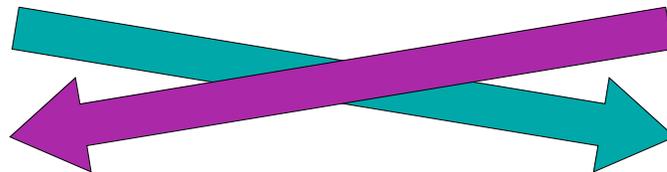
  If I don't love it, then it's not a puppy.

The Contrapositive

- The ***contrapositive*** of the implication

  If *P* is true, then *Q* is true

  is the implication

  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

*If **I store cat food inside**, then **raccoons won't steal it.***
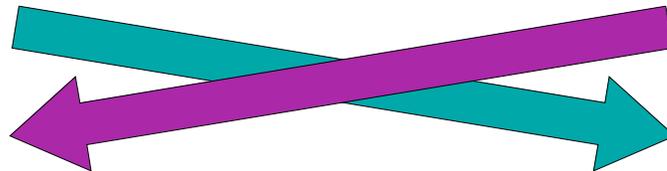
*If **raccoons stole the cat food**, then **I didn't store it inside**.*

To prove the statement

"if **P is true**, then **Q is true**,"

you can choose to instead prove the equivalent statement

"if **Q is false**, then **P is false**,"

if that seems easier.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Proof: We will prove the contrapositive of this statement

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Proof: We will prove the contrapositive of this statement.

This is a courtesy to the reader and says "heads up! we're not going to do a regular old-fashioned direct proof here."

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

What is the contrapositive of this statement?

if n^2 is even, then n is even.

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What is the contrapositive of this statement?

if n^2 is even, then n is even.

If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

What is the contrapositive of this statement?
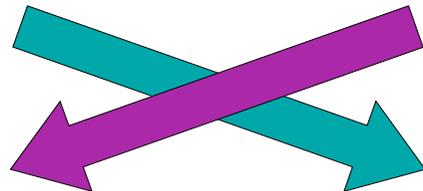
if n^2 **is even**, then n **is even.**

If n **is odd**, then n^2 **is odd.**

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that **if n is odd, then n^2 is odd**.

We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

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We know that n is odd, which means there is an integer k such that $n = 2k + 1$.

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We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$n^2 = (2k + 1)^2$$

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$$n^2 = (2k + 1)^2$$
$$= 4k^2 + 4k + 1$$

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We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$
\begin{aligned}
n^2 &= (2k + 1)^2 \\
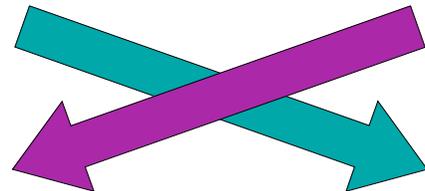&= 4k^2 + 4k + 1 \\
&= 2(2k^2 + 2k) + 1.
\end{aligned}
$$

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$$n^2 = (2k + 1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1.$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

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We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

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\begin{aligned}
n^2 &= (2k + 1)^2 \\
&= 4k^2 + 4k + 1 \\
&= 2(2k^2 + 2k) + 1.
\end{aligned}
$$

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We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$n^2 = (2k + 1)^2$$
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We kno
integer
us that

From th
(namely
means t
to show. ▪

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

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\begin{aligned}
n^2 &= (2k + 1)^2 \\
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\end{aligned}
$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. That means that n^2 is odd, which is what we needed to show. ∎

Biconditionals

- The previous theorem, combined with what we saw on Wednesday, tells us the following:

  For any integer n, if n is even, then n^2 is even.

  For any integer n, if n^2 is even, then n is even.

- These are two different implications, each going the other way.

- We use the phrase *if and only if* to indicate that two statements imply one another.

- For example, we might combine the two above statements to say

  for any integer n: n is even if and only if n^2 is even.

Proving Biconditionals

- To prove a theorem of the form

  P if and only if Q,

  you need to prove two separate statements.

  - First, that if P is true, then Q is true.
  - Second, that if Q is true, then P is true.

- You can use any proof techniques you'd like to show each of these statements.

  - In our case, we used a direct proof for one and a proof by contrapositive for the other.

What We Learned

- ***How do you negate formulas?***

  - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.

- ***What's a proof by contradiction?***

  - It's a proof of a statement P that works by showing that P cannot be false.

- ***What's an implication?***

  - It's statement of the form "if P, then Q," and states that if P is true, then Q is true.

- ***What is a proof by contrapositive?***

  - It's a proof of an implication that instead proves its contrapositive.

  - (The contrapositive of "if P, then Q" is "if not Q, then not P.")

Your Action Items

- ***Read "Guide to Office Hours," the "Proofwriting Checklist," and the "Guide to LaTeX."***
  - There's a lot of useful information there. In particular, be sure to read the Proofwriting Checklist, as we'll be working through this checklist when grading your proofs!
- ***Start working on PS1.***
  - At a bare minimum, read over it to see what's being asked. That'll give you time to turn things over in your mind this weekend.

Next Time

- ***Mathematical Logic***

  - How do we formalize the reasoning from our proofs?

- ***Propositional Logic***

  - Reasoning about simple statements.

- ***Propositional Equivalences***

  - Simplifying complex statements.

Appendix: Proving Implications by Contradiction

Proving Implications

- Suppose we want to prove this implication:

  If P is true, then Q is true.

- We have three options available to us:

  - ***Direct Proof:***

  - ***Proof by Contrapositive.***

  - ***Proof by Contradiction.***

Proving Implications

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        Assume P **is true**, then prove Q **is true**.

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    Assume *P* **is true**, then prove *Q* **is true**.

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    Assume *Q* **is false**, then prove that *P* **is false**.

  - **Proof by Contradiction.**

    … what does this look like?

Theorem: For any integer n, if n^2 is even, then n is even.

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What is the negation of our theorem?

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$$n = 2k + 1. \tag{1}$$

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Squaring both sides of equation (1) and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned} \tag{2}$$

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> ## The three key pieces:
>
> 1. Say that the proof is by contradiction.
> 2. Say what the negation of the original statement is.
> 3. Say you have reached a contradiction and what the contradiction entails.
>
> In CS103, please include all these steps in your proofs!

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  - **Proof by Contrapositive.**

    Assume **Q is false**, then prove that **P is false**.

  - **Proof by Contradiction.**

    Assume **P is true** and **Q is false**,
    then derive a contradiction.